MAT1332 - MAT1732
Solutions, Assignment #5

Question 1
[4 points] Put the following in standard form (i.e. in the form $z = a + ib$).
a) $(2+3i)(1+i)$ **b**) $|-4+2i|$ **c**) $2\,\overline{(-4+2i)}$ **d**) $\frac{1+i}{1-i}$

Solution:
a) $-1+5i$ **b**) $\sqrt{20}$ **c**) $-8-4i$ **d**) i

Question 2
[2 points] Put $z = 1+3i$ in its trigonometric form (i.e. in the form $z = |z|\,(\cos\theta + i\sin\theta)$.

Solution:
$|z| = \sqrt{1+3^2} = \sqrt{10}$, $\theta = \tan^{-1}(3/1) = \tan^{-1}(3) \approx 1.25$.

Question 3
[6 points] Find the eigenvalues and the eigenvectors of $A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix}$.

Solution:
Eigenvalues : Find when $(A-\lambda I)x = 0$ has non trivial solutions, i.e. when $\det(A-\lambda I) = 0$.

$$\det\begin{pmatrix} -2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 4 \\ 0 & 1 & 1-\lambda \end{pmatrix} = (-\lambda-2)\left((1-\lambda)^2-4\right) = -(\lambda+2)\left(\lambda^2-2\lambda+3\right) = ...$$

$$... = -(\lambda+2)(\lambda-3)(\lambda+1) = 0.$$

The eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = 3$ and $\lambda_3 = -1$

The eigenvectors u, v and w of A are found by solving the homogeneous systems $(A - \lambda_1 I)u = 0$, $(A-\lambda_2 I)v = 0$ and $(A-\lambda_3 I)w = 0$ respectively.

First eigenvector :

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The first component of the eigenvector u is a free variable. Parametrisation : $u_1 = t \in \mathbb{R}$. We have $u_2 = 0$ and $u_3 = 0$.

$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

With $t = 1$, we obtain $u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, one of the eigenvector associated with the eigenvalue $\lambda_1 = -2$.

Second eigenvector :

$$\left(\begin{array}{ccc|c} -5 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The third component of v is a free variable. Parametrisation : $v_3 = s \in \mathbb{R}$. We obtain the equations $v_1 = 0$, $v_2 - 2v_3 = 0$ and then that $v_2 - 2s = 0$. The solution can be written as

$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2s \\ s \end{pmatrix} = s \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$.

Note : With $s = 1$, we have $v = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, one of the eigenvector associated with the eigenvalue $\lambda_2 = 3$.

Third eigenvector :

$$\left(\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The third component of the eigenvector w is a free variable. Parametrisation : $w_2 = p \in \mathbb{R}$. We obtain the equations $w_1 = 0$, $w_2 + 2w_3 = 0$ and then that $w_2 + 2p = 0$. The solution can be written as

$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -2p \\ p \end{pmatrix} = p \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$.

With $p = 1$, we obtain $w = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$, one of the eigenvector associated with the eigenvalue $\lambda_3 = -1$.

Question 4

[2 points] Find the eigenvalues of $B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

Bonus : [2 points] Find the eigenvectors of B.

Solution:

Eigenvalues : Find when $(B - \lambda I)x = 0$ has non trivial solutions, i.e. when $\det(B - \lambda I) = 0$.

$$\det \begin{pmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 4 = \lambda^2 - 2\lambda + 5 = 0$$

quand $\lambda = 1 \pm 2i$.

The eigenvalues of B are $\lambda_1 = 1 - 2i$ and $\lambda_2 = 1 + 2i$.

Eigenvectors (Bonus) The eigenvectors u and v of B are found by solving the homogeneous systems $(B - \lambda_1 I)u = 0$ et $(B - \lambda_2 I)v = 0$ respectively.

$$\left(\begin{array}{cc|c} 2i & -2 & 0 \\ 2 & 2i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

The second component of u is a free variable. Parametrisation : $u_2 = t \in \mathbb{C}$. We obtain the equation $u_1 + iu_2 = 0$ and then $u_1 + it = 0$. The solution can be written as $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -it \\ t \end{pmatrix} = t \begin{pmatrix} -i \\ 1 \end{pmatrix}$, the eigenvector associated with the eigenvalue $\lambda_1 = 1 - 2i$ with t, a complex number (one could take t=1 for example, to work with a single eigenvector).

For the second eigenvector, we have

$$\left(\begin{array}{cc|c} -2 & -2 & 0 \\ 2 & -2i & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right).$$

The second component of v is a free variable. Parametrisation : $v_2 = s \in \mathbb{C}$. We obtain the equation $v_1 - iv_2 = 0$ and then $v_1 - it = 0$. The solution can be written as

$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} is \\ s \end{pmatrix} = s \begin{pmatrix} i \\ 1 \end{pmatrix}$, the eigenvector associated with the eigenvalue $\lambda_2 = 1 + 2i$ with s, a complex number (one could take s=1 for example, to work with a single eigenvector).

Question 5

[6 points] Let $x_{n+1} = Ax_n$ be a discrete dynamical system, with the initial state $x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 0.5 \\ 0 & 0.5 \end{pmatrix}$.

a) Why is A a *Markov* matrix?

b) Find the eigenvectors and the eigenvalues of A.

c) Find the equilibrium point (i.e. the eigenvector associated with the eigenvalue $\lambda = 1$) by choosing the parameter such that the sum of the components of the equilibrium point is equal to 1.

d) Write the initial state x_0 as a linear combination of the eigenvectors of A (with the equilibrium written as in **c**)).

e) Write the solution x_n by calculating $x_n = A^n x_0$.

Solution:

a) The sum of the elements of each column of A gives 1.

b)

Eigenvalues

$$\det \begin{pmatrix} 1-\lambda & 0.5 \\ 0 & 0.5-\lambda \end{pmatrix} = (1-\lambda)(0.5-\lambda) = 0.$$

$\lambda_1 = 1$ and $\lambda_2 = 0.5$ are the eigenvalues of A.

Eigenvectors To find u an eigenvector associated with $\lambda_1 = 1$, we solve

$$\left(\begin{array}{cc|c} 1-\lambda_1 & 0.5 & 0 \\ 0 & 0.5-\lambda_1 & 0 \end{array} \right) = \left(\begin{array}{cc|c} 0 & 0.5 & 0 \\ 0 & -0.5 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

u_1 is free, $u_1 = t$, $t \in \mathbb{R}$. $u_2 = 0$. Then $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. For $u_1 + u_2 = 1$, we choose $t = 1$.

To find v an eigenvector associated with $\lambda_2 = 0.5$, we solve

$$\left(\begin{array}{cc|c} 1-\lambda_2 & 0.5 & 0 \\ 0 & 0.5-\lambda_2 & 0 \end{array} \right) = \left(\begin{array}{cc|c} 0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{array} \right) \sim \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

v_2 is free, $v_2 = s$, $s \in \mathbb{R}$. $v_1 + s = 0$. Then $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. For simplicity, we choose $s = 1$.

c) The equilibrium point is $\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

d) Finding such a linear combination is equivalent to the problem of : finding the coefficients k_1 and k_2 such that

$$x_0 = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

With $x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, it means that we have to solve the following linear system of equations :

$$\begin{aligned} k_1 - k_2 &= 0 \\ k_2 &= 1 \end{aligned},$$

which can be written as an augmented matrix that we can reduce with the Gauss-Jordan method.

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right)$$

We have finally that $k_1 = 1$ and $k_2 = 1$ so that the linear combination is

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

e) $x_n = A^n \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right) = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + A^n \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (1)^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1/2)^n \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$x_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (1/2)^n \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Note : $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector of A associated with the eigenvalue $1/2$ so

$$A \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (1/2) \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Multiplying the last equation by A (by the left), we get

$$AA \begin{pmatrix} -1 \\ 1 \end{pmatrix} = A(1/2) \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (1/2)A \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (1/2)^2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

So if we apply n times A (by the left) to $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, we get

$$A^n \begin{pmatrix} -1 \\ 1 \end{pmatrix} = (1/2)^n \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Same thing for the other eigenvector.